T6A042

"Reliability – Effects of Proof Testing

1 Contents

2 Revision History

Version	Date	Author	Comments
0.1	18/01/2022	R. Martin	Draft release for public comment.
0.2	28/02/2022	R. Martin	Title changed and document restructured to fully cover the topics of synchronised and staggered proof testing. Added wider context of reliability and references.
0.3	25/01/23	R. Martin	Minor corrections to text.
0.4	16/08/2023	R. Martin	Minor corrections to text. Foreword and Executive Summary updated. Changed algebraic expansion to KooL to save confusion with MooN
0.5	17/01/2024	R. Martin	Major update to add section on the effect of synchronised testing and mission time replacement.
1.0	26/01/2024	R. Martin	First issue.

3 Introduction / Foreword

This document covers the topic of proof test effects in relation to functional safety and is part of a series of documents linking together to support a reliability calculation tool.

There are four documents in the series:
1. Reliability and Availability
2. Effects of Proof Testing
3. Fault Tolerant Systems
4. Staggered Proof Testing Coefficients

Reliability and Availability explores the basic mathematics of reliability and explains:
- What is meant by constant failure rate;
- The effect of parallel and series networks;
- The relationship between λ and MTBF;
- The importance of repairable systems and Availability (as an average over time);
- The time average likelihood of being in a failed state (so called PFD_{AV});
- The other terms in common use for diagnosed and undiagnosed failures;
- The differing effects of diagnosed and undiagnosed failures;
- The effects of common proof testing regimes on multiple failures;
- The effects of common cause failures
- Simple and complex redundancy;
- Conditional Probability;
- Estimating reliability from data.

This document is the second in the series. It looks specifically at the effect of proof testing and the strategy employed. Proof testing scenarios considered are:
- synchronous proof test (where all components in parallel are proof tested in the same task at time interval, T)
- staggered proof test (where the proof testing is on each component is carried out at time intervals, T, but where they are evenly spaced in time.

Each of the above has a distorting effect on PFD^i such that $PFD^i \neq (PFD)^i$
The study of both strategies returns *Test Coefficients* which are used to compensate for the above inequality.

This document expands on some of the issues already discussed by looking at further at the distorting effects of proof testing strategy on the algebra but not the sub-sets for:

- Synchronous proof testing
- Staggered proof testing
-

It should be noted that:

- synchronous proof testing has a detrimental effect such that $PFD^i > (PFD)^i$
- staggered proof testing has a beneficial effect such that $PFD^i < (PFD)^i$

The generation of coefficients for staggered proof testing is complex and thus there is an additional document *Staggered Proof Testing Coefficients* dedicated to their development.

The coefficients demonstrate that, the more channels, the more staggered proof testing appears to provide a reliability advantage over synchronous testing. Also, if a channel fault is found during a staggered test, the approach is to then test further to identify if common cause failures have occurred – this approach reduces the common cause effect by a factor of N.

Because of the interests in functional safety, the theory is related to safety wherever possible. It should however be understood that reliability and availability are broader topics. So, although it is related to functional safety, it is not a 'safety only' subject and the maths derived is just as applicable to reliability in general.

There are other models available - e.g. ISA 84 Part 2 [3], SINTEF PDS method [4]. Some are more comprehensive than others and all have limitations. IEC 61508 [1] stresses the importance that the analyst understands the techniques and the limitations of any underlying hypotheses. This series of documents is written with that in mind. Note: The standard itself uses a complex approach where 'mean channel downtime' is treated as critical and often causes confusion in what turn out to be 'self-cancelling' formulae. There is no reasoning offered for this approach and, in this respect, the authors feel the standard fails its own criteria. These documents use a more traditional approach.

4 Executive Summary

The development of this series of documents came as a result of *The 61508 Association* (T6A) setting up a working group (WG) to produce good practice guidance on 'SIL Assessment' (the assessment of the ability of a system to perform a required safety function with the required integrity).

The history of the development is as follows:

- T6A set up WG15 to produce a good practice guide for 'SIL Assessment'.
- It became apparent that a spreadsheet would be the most suitable tool to use because of its ability with computational calculations and the ease of access and familiarity to most people.
- It also became apparent that the spreadsheet needed a 'built in' reliability calculator so that all important reasoning could be separated from number crunching but also that 'verification' in any instance of use would be confined to the reasoning and the appropriate use of the calculator rather than the calculator itself. So, it was decided to create the 'built in' calculator.
- Before creating the calculator, it became necessary to produce the formulae upon which the calculator would be based.
- Reliability is taught at many higher educational establishments and there is much information on safety related systems calculations in circulation. However, the authors were unable to find a source that pulled it all together into general formulae. A document entitled 'Fault Tolerant Systems' was therefore created covering the development of the necessary formulae for calculating the failure rate and the probability of failure for so called 'MooN' fault tolerant systems.
- The formulae developed catered for diagnosed and undiagnosed failures, distortion due to synchronous proof testing and common cause failures.
- However, when the document was being verified, it became clear that verifiers needed some further explanation of the maths and (importantly) the development of the necessary terminology.
- Over time, it emerged that limited proof test coverage was becoming an issue of interest (especially to regulators). It also emerged that staggered proof testing for higher order systems gave considerable 'on paper' benefits. So, it was decided to add these two features to the calculator.
- As a result, three further documents were considered necessary:
    - One that covered the theory from first principles (now entitled Reliability and Availability).
    - One that covered the distorting effects of synchronous and staggered proof testing on the calculations (now entitled Effects of Proof Testing).
    - Because finding the distorting effects of staggered testing proved to be quite complex (a mixture of analytical and numerical techniques were used) it was

decided to make the deduction of the staggered testing coefficients into separate documents (now entitled Staggered Proof Testing Coefficients).

The formulae have now been developed from first principles and the spreadsheet calculator produced. The documents and the calculator have been independently verified.

5 Terminology

f	General term for 'fault tolerance' – i.e. for simple redundancy, the number of failed devices a system can tolerate and still perform its function. Note: r is the general term for the number of survivors required for a system to perform its function.
F	Probability of failure (normally a function of time). Note: this has the same meaning at PFD (probability of failure on demand).
MT	Mission Time (for use with residual failures)
MTBF	Mean time before failure. $\mathrm{MTBF} = 1/\lambda$ (for constant λ)
MTTR	Mean time to restore.
PFD	Probability of failure on demand. Notes: • this has the same meaning as F (probability of failure). • This is sometimes used in the text as shorthand for PFD_{AV}.
PFD_{AV}	Time average of PFD.
PFD_D	PFD for diagnosed failures for single channel / device. $PFD_D = PFD_D^1 = \left((1-\beta_D)\lambda dd.MTTR\right)$
PFD_R	PFD for residual failures for single channel / device. $PFD_R = PFD_R^1 = \left(\frac{(1-\beta_R)\lambda drMT}{2}\right)$
PFD_U	PFD for undiagnosed failures for single channel / device. $PFD_U = PFD_U^1 = \left(\frac{(1-\beta_U)\lambda duT}{2}\right)$
PFD_D^k	PFD for diagnosed failures for k channels / devices $PFD_D^k = (PFD_D)^k$
PFD_R^k	PFD for residual failures for k channels / devices $PFD_R^k \neq (PFD_R)^k$ due to test regime
PFD_U^k	PFD for undiagnosed failures for k channels / devices $PFD_U^k \neq (PFD_U)^k$ due to replacement regime
PFD^N	PFD rolled up for all failures for N channels / devices (including common causes)
R	Probability of survival (normally a function of time).
S	Used as a suffix to represent attributes of a system. E.g. F_S is used to represent probability of system failure.
T	Proof test interval.

β	Beta factor – general term for fraction of failures which affect all channels / devices.
β_D	Beta factor specific to diagnosed failures
β_R	Beta factor specific to residual failures
β_U	Beta factor specific to undiagnosed failures
λ	General term for underlying failure rate – a function of time that represents the failure rate 'given that there is no current failure'. This document assumes it is a constant in time. Note: this is not the same as $\dot{\mathrm{F}}(\mathrm{t})$ (which is the failure rate not assuming current survival).
λ_d	General term for diagnosed failure rate – i.e. failure that is automatically revealed.
λ_u	General term for undiagnosed failure rate.
λ_{dd}	Dangerous diagnosed failure rate.
λ_{dr}	Dangerous residual failure rate – i.e. dangerous failure rate that is not automatically revealed or revealed by periodic proof test.
λ_{du}	Dangerous undiagnosed failure rate.

6 Reliability Model

The accepted model (including that adopted by IEC 61508) is that of random hardware failures and constant failure rates in the throughout the useful life. Whilst this is a useful approximation in estimating reliability, it should be understood that reliability is not an exact science and approaches to modelling are still evolving.

Industrial databases of reliability statistics (such as OREDA) are often used in modelling the expected failure rates of complex systems. In practice, such databases tend to be conservative because they often account for failures wider than those of random hardware failures. This tends to lead to conservative claims (which is probably where we would like them to be in matters of safety).

However, caution is advised. Reliability of components of similar type can vary depending on the source. Stress factors in the installed environment can lead to considerable variation (i.e. it is not unusual to see variances of up to a factor of 3 either side of the norm.

The calculations described in this guideline may be applied to estimate the probability of failure for electrical, mechanical, pneumatic or hydraulic devices, but the precision is limited by the extent to which users can achieve reasonably consistent failure performance. The performance of equipment should be continually kept under review and maintenance practices and associated calculations modified to take account of findings.

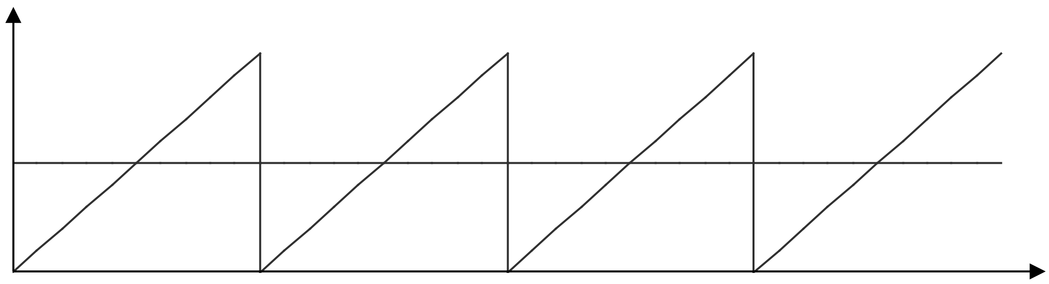
The reader is advised to read as widely as practicable in order to understand the pitfalls of over-reliance on unrealistic assumptions. Books such as *Reliability, Maintainability and Risk* by Dr David J Smith [5] and papers such as *New approach to SIL verification* by Mirek Generowicz [6] make very useful reading in setting the overall context.

There are many other sources of information and guidance for reliability and availability, for example simplified formulas via ISA-TR84.00.02 and VDI/VDE 2180 Part 3 or IEC 61508-6:2010 Annex B (informative) for examples of more complete formulas.

7 Synchronised Proof Testing in Simple Redundancy

Each time a device is tested it will either be found to be working or it will be repaired. The effect on the probability of failure as a function of time is shown below. At each proof test, the probability of failure is 'reset' to 0. This results in the 'saw tooth' type function.

Note: In safety systems, we refer to the *probability of failure on demand* (PFD) but this is no different from F(t). In particular, we refer to the average probability of failure on demand (PFD_{AV}) because this becomes a very useful measure when considering overall risk.



We can see from the above that the average over time is the same as the average over one proof test interval.

$$PFD_{AV} = \frac{1}{T}\int_0^T \lambda t dt = \frac{\lambda}{T}\frac{T^2}{2} = \frac{\lambda T}{2}$$

$$PFD_{AV} = \frac{\lambda T}{2}$$

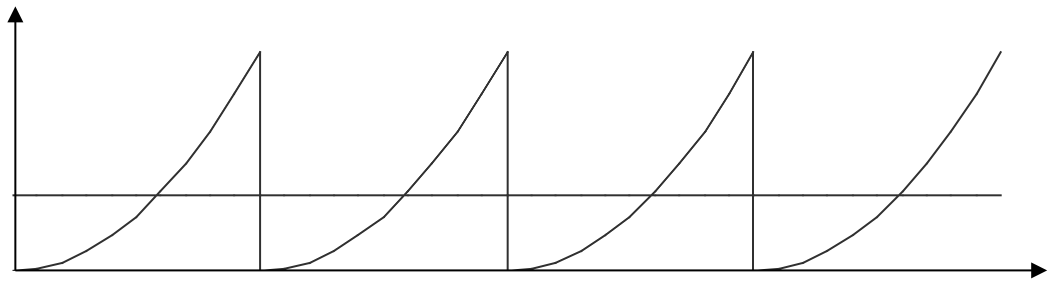
We can write this as:

$$PFD_{1oo1} = \frac{\lambda T}{2}$$

Note: The above ignores the time out of service during which an item discovered in the failed condition is 'under repair'.

7.1 1oo2 simple redundancy

If we assume that testing is synchronised (i.e. both devices are tested at the same time), the PFD_{AV} is derived as follows.



The average of a cycle is given by:

$$PFD_{AV} = \frac{1}{T}\int_0^T \lambda^2 t^2 dt = \frac{\lambda^2}{T}\frac{T^3}{3} = \frac{\lambda^2 T^2}{3}$$

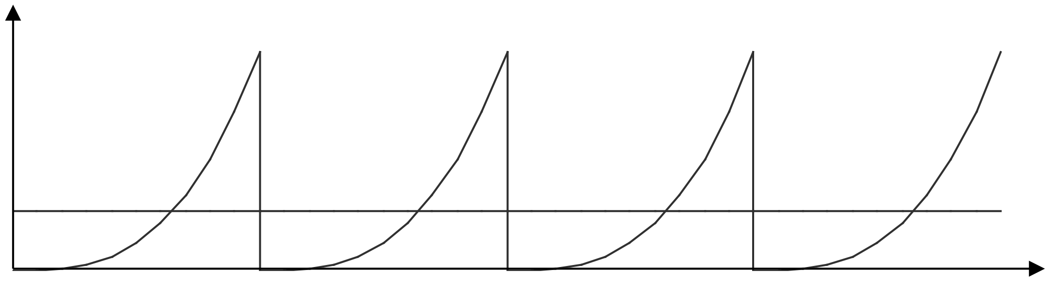
$$PFD_{AV} = \frac{\lambda^2 T^2}{3}$$

So, for synchronised proof testing there is a degradation factor of 4/3, i.e.:

$$PFD_{1oo2} = \frac{4}{3} PFD_{1oo1}^2$$

7.2 1oo3 simple redundancy

If we assume that testing is synchronised, the resulting PFD_{AV} is derived as follows.



The average of a cycle is given by:

$$PFD_{AV} = \frac{1}{T}\int_0^T \lambda^3 t^3 dt = \frac{\lambda^3}{T}\frac{T^4}{4} = \frac{\lambda^3 T^3}{4}$$

So, for synchronised proof testing there is a degradation factor of 8/4, i.e.:

$$PFD_{1oo3} = 2PFD_{1oo1}^3$$

In this latter case, the PFD average of the system is twice what we would get by taking the cube of the simplex PFD_{AV}.

7.3 1ooN simple redundancy

The effect on PFD_{AV} of synchronised proof testing on simple redundancy is summarised in the following table. Note: here 1ooN represents 1 out of N for system survival.

To survive	PFD_{AV}
1oo1	PFD_{1oo1}
1oo2	$4/3(PFD_{1oo1})^2$
1oo3	$8/4(PFD_{1oo1})^3$
1oo4	$16/5(PFD_{1oo1})^4$
1oo5	$32/6(PFD_{1oo1})^5$
1ooN	$(2^N/N+1)(PFD_{1oo1})^N$

Trying the formula for 2oo2 to fail, $PFD_{AV} = 4/3(PFD_{1oo1})^2$, i.e.:

$$PFD_{AV} = \frac{4}{3}\frac{\lambda^2 T^2}{4}$$

Note: For 2oo3, there are 3 possible combinations of 2oo2 to fail so the PFD_{AV} is 3 times that:

$$PFD_{2oo3} = \lambda^2 T^2$$

So 2oo3 has a worse safety performance than 2oo2 to fail!

The reasons we commonly use 2oo3 are:

- it reduces spurious system failure rate and
- it allows additional discrepancy checking.

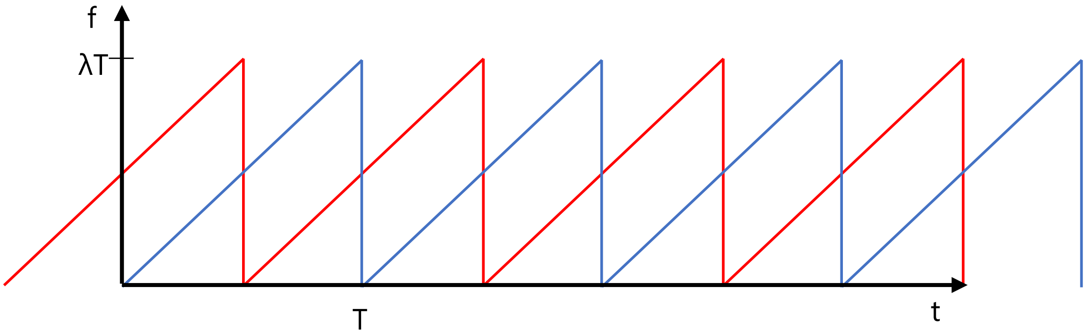
8 Staggered Proof Test in Simple Redundancy

Staggered proof testing has a different effect on PFD_{AV}. Whereas, with synchronous testing, the effect for 1ooN system is a PFD_{AV} which is worse than $(PFD_{1oo1})^N$, staggered proof testing has the opposite effect.

Note: It would be very unusual for inputs of a SIF function to be tested 'on rotation'. Given that it leads to some quite complex algebra, it would be possible to accept that proof testing on rotation (staggered proof testing) is a possibility but not to study the development of the relevant coefficients.

8.1 1oo2 simple redundancy (2oo2 to fail)

If we assume that testing is staggered evenly, the PFD_{AV} is derived as follows:



The red and blue lines represent the failure probability of the two channels. The failure probability of the system is the product of the two. We can see that the system failure probability repeats each period of T/2.

The average can therefore be found by integrating over any period T/2 and dividing by the period. Starting at T=0:

$$F(t) = \lambda t.(\lambda t + \frac{\lambda T}{2})$$

$$F(t) = \lambda^2 (t^2 + \frac{T}{2} t)$$

$$PFD_{AV} = F_{AV} = \frac{1}{T/2} \int_0^{T/2} F(t)\, dt$$

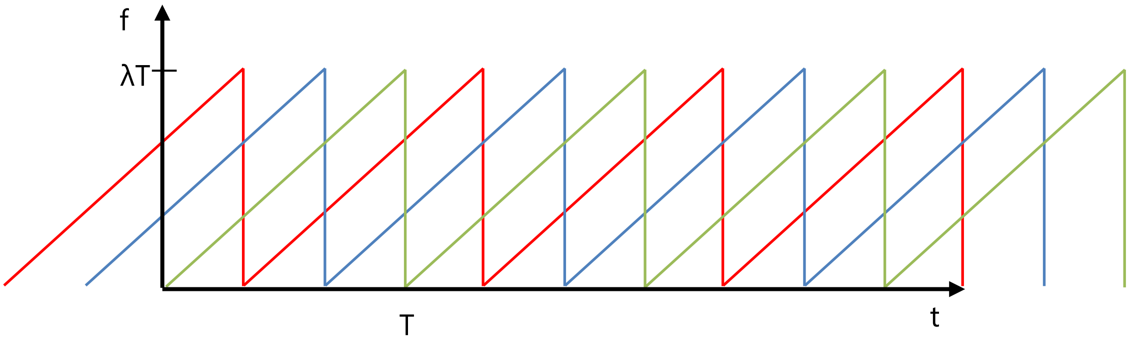
$$PFD_{AV} = \frac{1}{T/2} \int_0^{T/2} \lambda^2 \left(t^2 + \frac{T}{2} t\right) dt$$

$$PFD_{AV} = \frac{\lambda^2}{T/_2}\left[\frac{t^3}{3} + \frac{T}{2}.\frac{t^2}{2}\right]_0^{T/_2}$$
$$PFD_{AV} = \frac{\lambda^2}{T/_2}\left(\frac{T^3}{3.2^3} + \frac{T^3}{2.2^3}\right)$$
$$PFD_{AV} = \lambda^2 T^2\left(\frac{1}{3.2^2} + \frac{1}{2.2^2}\right)$$
$$PFD_{AV} = \frac{\lambda^2 T^2}{2^2}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{6}.\frac{\lambda^2 T^2}{2^2}$$

$$PFD_{AV} = \frac{5}{6} PFD_{1oo1}{}^2$$

8.2 1oo3 simple redundancy

If we assume that testing is staggered evenly, the resulting PFD_{AV} is derived as follows.



The red, blue and green lines represent the different channels and the combined system failure is the product of the three. We can see that the system failure repeats each period of T/3.

The average is found by integrating over the period and dividing by the period. In each period of T/3:

$$F(t) = \lambda t.\left(\lambda t + \frac{\lambda T}{3}\right).\left(\lambda t + \frac{2\lambda T}{3}\right)$$

$$F(t) = \lambda^3\left(t^3 + 3\left(\frac{T}{3}\right)t^2 + 2\left(\frac{T}{3}\right)^2 t\right)$$

$$PFD_{AV} = F_{AV} = \frac{1}{T/_3}\int_0^{T/_3} F(t)\,dt$$

$$PFD_{AV} = \frac{1}{T/_3}\int_0^{T/_3} \lambda^3\left(t^3 + 3\left(\frac{T}{3}\right)t^2 + 2\left(\frac{T}{3}\right)^2 t\right)dt$$

$$PFD_{AV} = \frac{\lambda^3}{T/_3}\left[1.\frac{1}{4}t^4 + 3.\frac{1}{3}\left(\frac{T}{3}\right)t^3 + 2.\frac{1}{2}\left(\frac{T}{3}\right)^2 t^2\right]_0^{T/_3}$$

$$PFD_{AV} = \frac{\lambda^3}{T/_3}\left(1.\frac{1}{4}\left(\frac{T}{3}\right)^4 + 3.\frac{1}{3}\left(\frac{T}{3}\right)^4 + 2.\frac{1}{2}\left(\frac{T}{3}\right)^4\right)$$

$$PFD_{AV} = \frac{\lambda^3 T^3}{2^3}.\frac{2^3}{3^3}\left(1 \times \frac{1}{4} + 3 \times \frac{1}{3} + 2 \times \frac{1}{2}\right)$$

$$PFD_{AV} = \frac{\lambda^3 T^3}{2^3}\left(\frac{2^3}{3^3}.\frac{9}{4}\right)$$

$$PFD_{AV} = \left(\frac{2}{3}\right)\frac{\lambda^3 T^3}{2^3}$$

$$PFD_{AV} = \frac{2}{3} PFD_{1oo1}{}^3$$

8.3 1ooN simple redundancy

The general case is slightly complicated because the number pattern is difficult to generate.

$$F_N(t) = \lambda t\left(\lambda t + \frac{\lambda T}{N}\right)\left(\lambda t + \frac{2\lambda T}{N}\right)\left(\lambda t + \frac{3\lambda T}{N}\right)\dots\dots$$

$$F_N(t) = \prod_{i=0}^{N}\left(\lambda t + \frac{i\lambda T}{N}\right)$$

To make the algebraic development a little easier to follow, make the following substitution.

$$a = \lambda t$$
$$b = \frac{\lambda T}{N}$$

$$F_1(t) = a$$

$$F_2(t) = a(a + b) = a^2 + ab$$

The coefficients for F_2 are 1, 1.

$$F_3(t) = a(a + b)(a + 2b)$$
$$F_3(t) = (a^2 + ab)(a + 2b)$$
$$F_3(t) = a^3 + a^2 b + 2a^2 b + 2ab^2$$
$$F_3(t) = a^3 + 3a^2 b + 2ab^2$$

The coefficients for F_3 are 1, 3, 2.

$$F_4(t) = (a^3 + 3a^2b + 2ab^2)(a + 3b)$$
$$F_4(t) = (a^4 + 3a^3b + 2a^2b^2 + 3a^3b + 9a^2b^2 + 6ab^3)$$
$$F_4(t) = (a^4 + 6a^3b + 11a^2b^2 + 6ab^3)$$

The coefficients for F_4 are 1, 6, 11, 6.

By using the expansion method above, the coefficients for N can be developed (although the number pattern is quite complex).

	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0	0	0
4	1	6	11	6	0	0	0	0	0	0
5	1	10	35	50	24	0	0	0	0	0
6	1	15	85	225	274	120	0	0	0	0
7	1	21	175	735	1624	1764	720	0	0	0
8	1	28	322	1960	6769	13132	13068	5040	0	0
9	1	36	546	4536	22449	67284	118124	109584	40320	0
10	1	45	870	9450	63273	269325	723680	1172700	1026576	362880

For the case of $1oo4$:

$$PFD_{AV} = \frac{1}{T/_4}\int_0^{T/_4} 1(\lambda t)^4\left(\frac{\lambda T}{4}\right)^0 + 6(\lambda t)^3\left(\frac{\lambda T}{4}\right)^1 + 11(\lambda t)^2\left(\frac{\lambda T}{4}\right)^2 + 6(\lambda t)^1\left(\frac{\lambda T}{4}\right)^3 dt$$

$$PFD_{AV} = \frac{1}{T/_4}\left[1(\lambda t)^4\left(\frac{\lambda T}{4}\right)^0 + 6(\lambda t)^3\left(\frac{\lambda T}{4}\right)^1 + 11(\lambda t)^2\left(\frac{\lambda T}{4}\right)^2 + 6(\lambda t)^1\left(\frac{\lambda T}{4}\right)^3\right]_0^{T/_4}$$

$$PFD_{AV} = \frac{\lambda^4}{T/_4}\left[1(t)^4\left(\frac{T}{4}\right)^0 + 6(t)^3\left(\frac{T}{4}\right)^1 + 11(t)^2\left(\frac{T}{4}\right)^2 + 6(t)^1\left(\frac{T}{4}\right)^3\right]_0^{T/_4}$$

$$PFD_{AV} = \frac{\lambda^4}{T/_4}\left[1\times\left(\frac{1}{5}\right)\left(\frac{T}{4}\right)^5\left(\frac{T}{4}\right)^0 + 6\times\left(\frac{1}{4}\right)\left(\frac{T}{4}\right)^4\left(\frac{T}{4}\right)^1 + 11\times\left(\frac{1}{3}\right)\left(\frac{T}{4}\right)^3\left(\frac{T}{4}\right)^2 + 6\times\left(\frac{1}{2}\right)\left(\frac{T}{4}\right)^2\left(\frac{T}{4}\right)^3\right]$$

$$PFD_{AV} = \frac{\lambda^4T^4}{4^4}\left[1\times\left(\frac{1}{5}\right) + 6\times\left(\frac{1}{4}\right) + 11\times\left(\frac{1}{3}\right) + 6\times\left(\frac{1}{2}\right)\right]$$

$$PFD_{AV} = \left(\frac{\lambda T}{2}\right)^4 . \left(\frac{2}{4}\right)^4 \left[1 \times \left(\frac{1}{5}\right) + 6 \times \left(\frac{1}{4}\right) + 11 \times \left(\frac{1}{3}\right) + 6 \times \left(\frac{1}{2}\right)\right]$$

For the general case of $1ooN$:

$$PFD_{AV} = \left(\frac{\lambda T}{2}\right)^N . \left(\frac{2}{N}\right)^N . \left(\frac{St_{N,1}}{N+1} + \frac{St_{N,2}}{N} + \frac{St_{N,3}}{N-1} + \frac{St_{N,4}}{N-2} + \cdots\right)$$

Therefore:

$$PFD_{AV} = PFD_{1oo1}{}^N . \left(\frac{2}{N}\right)^N . \left(\frac{St_{N,1}}{N+1} + \frac{St_{N,2}}{N} + \frac{St_{N,3}}{N-1} + \frac{St_{N,4}}{N-2} + \cdots\right)$$

$$PFD_{AV} = PFD_{1oo1}{}^N . \left(\frac{2}{N}\right)^N \sum_{i=1}^{N} \left(\frac{St_{N,i}}{N+2-i}\right)$$

Where $St_{N,i}$ is the coefficient given in row N and column i in the table above.

For example, for $N = 5$, the PFD$_{AV}$ is found to be:

$$PFD_{AV} = \frac{\lambda^5 T^5}{2^5} . \frac{2^5}{5^5} \left(1 \times \frac{1}{6} + 10 \times \frac{1}{5} + 35 \times \frac{1}{4} + 50 \times \frac{1}{3} + 24 \times \frac{1}{2}\right)$$
$$PFD_{AV} = \frac{\lambda^5 T^5}{2^5} . \frac{2^5}{5^5} \left(39 \frac{7}{12}\right)$$
$$PFD_{AV} = \frac{152}{375} \frac{\lambda^5 T^5}{2^5}$$
$$PFD_{AV} = \frac{152}{375} PFD_{1oo1}{}^5$$

9 Standard Configurations

Our common configurations are 1oo1 to fail (simplex), 2oo2 to fail (duplex) and 2oo3.

Both 1oo1 to fail and 2oo2 to fail are deal with above. That leaves the case for 2oo3 to solve explicitly.

9.1 1oo1 to fail

For 1oo1 to fail, the we would expect for both synchronised and staggered testing for the result to be identical – i.e. there is no difference.

$$PFD_{AV} = PFD_{1oo1} = \frac{\lambda T}{2}$$

Using the formula for staggered testing, we get:

$$PFD_{AV} = PFD_{1oo1}{}^{N}.\left(\frac{2}{N}\right)^{N}\sum_{i=1}^{N}\left(\frac{St_{N,i}}{N+2-i}\right)$$

Where

$$N = 1$$

$$PFD_{AV} = PFD_{1oo1}.2\sum_{i=1}^{1}\left(\frac{St_{1,i}}{1+2-i}\right)$$

$$PFD_{AV} = PFD_{1oo1}.2.\frac{1}{2}$$

$$PFD_{AV} = PFD_{1oo1}$$

As we would expect.

9.2 2oo2 to fail

For 2oo2 to fail, we would expect a difference.

9.2.1 Synchronised Testing

$$PFD_{AV} = \frac{2^{N}}{N+1}PFD_{1oo1}{}^{N}$$

Where $N = 2$

$$PFD_{2oo2} = \frac{2^{2}}{2+1}PFD_{1oo1}{}^{2}$$

$$PFD_{2oo2} = \frac{4}{3}PFD_{1oo1}{}^{2}$$

9.2.2 Staggered Testing

$$PFD_{AV} = PFD_{1oo1}{}^{N}.\left(\frac{2}{N}\right)^{N}\sum_{i=1}^{N}\left(\frac{St_{N,i}}{N+2-i}\right)$$

Where $N = 2$

$$PFD_{2oo2} = PFD_{1oo1}{}^{2}.\left(\frac{2}{2}\right)^{2}\sum_{i=1}^{2}\left(\frac{St_{2,i}}{4-i}\right)$$

$$PFD_{2oo2} = PFD_{1oo1}{}^{2}.\left(\frac{1}{3}+\frac{1}{2}\right)$$

$$PFD_{2oo2} = \frac{5}{6}PFD_{1oo1}{}^{2}$$

As above.

9.3 2oo3 to fail

2oo3 is an example of a voted system which is not covered by the above general $1ooN$ formula.

9.3.1 Synchronised Testing

In a 2oo3 configuration, if 2 of the 3 fail then the system fails. There are 3 possible combinations that will give a pair of failures, where each pair has the same probability of failure. The probability of failure of a pair is the same as for the 2oo2 to fail configuration.

Therefore:

$$PFD_{2oo3} = 3\frac{2^N}{N+1}PFD_{1oo1}{}^N$$

Where

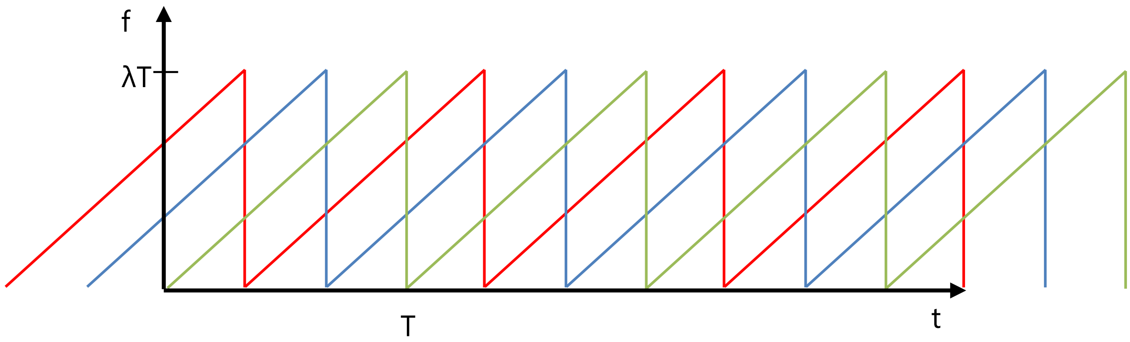
$$N = 2$$

$$PFD_{2oo3} = 3\frac{2^2}{2+1}PFD_{1oo1}{}^2$$

$$PFD_{2oo3} = 4PFD_{1oo1}{}^2$$

9.3.2 Staggered Testing

If we assume that we have 3 components (A, B and C), the system fails if A and B fail, B and C fail or A and C fail. We therefore need to look at the probability of each pair failing.



If we look at any pair in the above, we see they are not evenly distributed in time. But the period between them is always T/3. So, whichever pair we are considering will always have the same PFD_{AV}.

Let's take the green and red as a pairing in look at the average over the period T.

$$F(t) = \lambda t.\left(\lambda t + \frac{2\lambda T}{3}\right) for\ [t: 0, T/3]$$

$$F(t) = \lambda t.\left(\lambda t - \frac{\lambda T}{3}\right) for\ [t: T/3, T]$$

This leads to:

$$PFD_{2oo3} = \frac{2}{3}\lambda^2 T^2$$

$$PFD_{2oo3} = \frac{8}{3} PFD_{1oo1}{}^{2}$$

Refer to the *Staggered Proof Testing Coefficients* document for derivation.

9.4 **Summary**

For general MooN configurations, the effect of staggered proof testing is more challenging. The reason for this is that it is dependent on where the selection comes in the testing cycle.

Clearly staggering proof testing makes a considerable improvement to the system reliability over that of synchronised.

For higher values of M and N, the complexity increases considerably and. for that reason, this topic is dealt with separately in the section on complex redundancy (below).

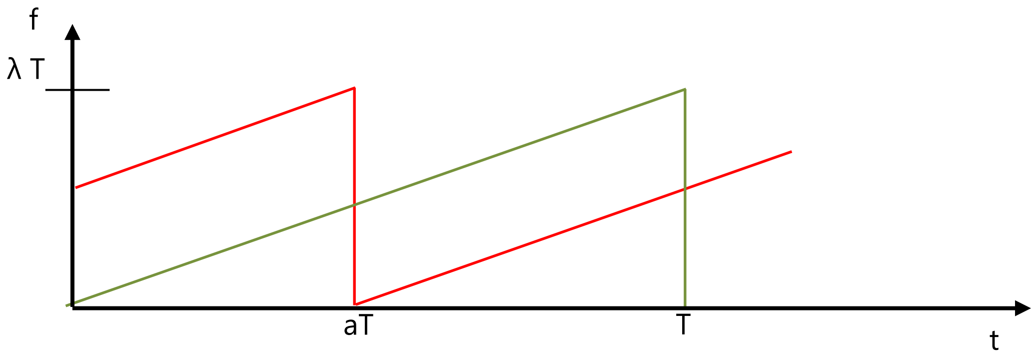
10 Staggered Proof Testing in Complex Redundancy

The latter cases of the standard configurations shown in the previous section are complex redundancy although they are relatively trivial examples.

In this section, we are going to try to generate a way of looking at the general case of MooN failures.

10.1 **Failure of 2 Items**

Consider the following graph



In the above diagram, there are two items with staggered proof test intervals shown by the 2 colours.

$F(t)_{green} = \lambda t$ [0,T]
$F(t)_{red} = \lambda t + (1-a)\lambda T$ [0,aT]
$F(t)_{red} = \lambda t + (-a)\lambda T$ [aT,T]

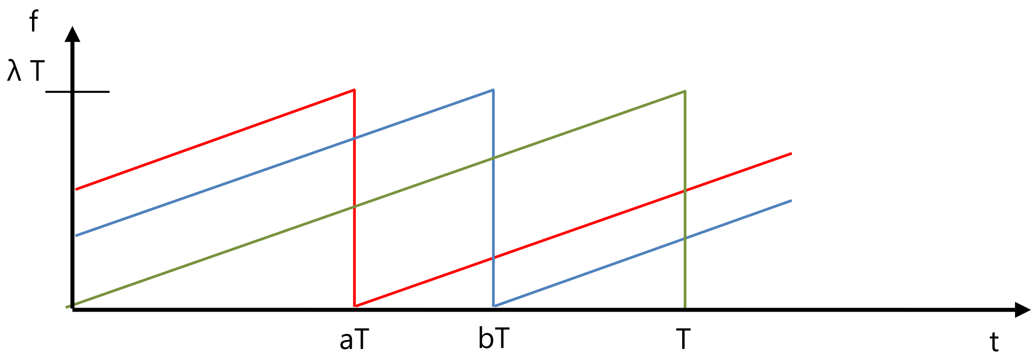
The joint probability of failure is given by:
$F(t) = \lambda t(\lambda t + (1-a)\lambda T) = \lambda^2(t^2 + (1-a)Tt)$ [0,aT]
$F(t) = \lambda t(\lambda t + (-a)\lambda T) = \lambda^2(t^2 + (-a)Tt)$ [aT,T]

$$PFD_{AV} = \frac{\lambda^2}{T}\left(\int_0^{aT} t^2 + (1-a)T.t\,dt + \int_{aT}^{T} t^2 + (-a)T.t\,dt\right)$$

$$PFD_{AV} = \frac{\lambda^2}{T}\left(\left[\frac{t^3}{3}\right]_0^{aT} + (1-a)T\left[\frac{t^2}{2}\right]_0^{aT} + \left[\frac{t^3}{3}\right]_{aT}^{T} + (-a)T\left[\frac{t^2}{2}\right]_{aT}^{T}\right)$$

10.2 **Failure of 3 Items**

Consider the following graph



In the above diagram, there are three items with staggered proof test intervals shown by the 3 colours.
$F(t)_{green} = \lambda t$ [0,T]

$F(t)_{red} = \lambda t + (1-a)\lambda T$ 0,aT]
$F(t)_{red} = \lambda t + (-a)\lambda T$ [aT,T]

$F(t)_{blue} = \lambda t + (1-b)\lambda T$ [0,bT]
$F(t)_{blue} = \lambda t + (-b)\lambda T$ [bT,T]

Note: $b > a$

The joint probability of failure is given by

$F(t) = \lambda t(\lambda t + (1-a)\lambda T)(\lambda t + (1-b)\lambda T)$ [0,aT]
$F(t) = \lambda t(\lambda t + (-a)\lambda T)(\lambda t + (1-b)\lambda T)$ [aT,bT]
$F(t) = \lambda t(\lambda t + (-a)\lambda T)(\lambda t + (-b)\lambda T)$ [bT,T]

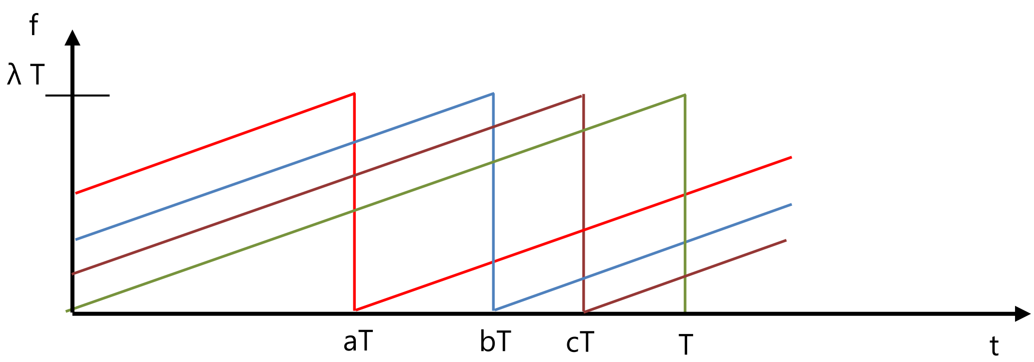
$F(t) = \lambda^3(t^3 + ((1-a)+(1-b))T.t^2 + ((1-a)(1-b))T^2.t)$ [0,aT]
$F(t) = \lambda^3(t^3 + ((-a)+(1-b))T.t^2 + ((-a)(1-b))T^2.t)$ [aT,bT]
$F(t) = \lambda^3(t^3 + ((-a)+(-b))T.t^2 + ((-a)(-b))T^2.t)$ [bT,T]

$$\begin{aligned} PFD_{AV} = {} & \frac{\lambda^3}{T}\left(\int_0^{aT} t^3 + ((1-a)+(1-b))T.t^2 + \big((1-a)(1-b)\big)T^2.t\,dt\right) \\ & + \frac{\lambda^3}{T}\left(\int_{aT}^{bT} t^3 + ((-a)+(1-b)).t^2 + \big((-a)(1-b)\big)T^2.t\,dt\right) \\ & + \frac{\lambda^3}{T}\left(\int_{bT}^{T} t^3 + ((-a)+(-b))T.t^2 + \big((-a)(-b)\big)T^2.t\,dt\right) \end{aligned}$$

$$\begin{aligned} PFD_{AV} = {} & \frac{\lambda^3}{T}\left[\frac{t^3}{4} + ((1-a)+(1-b))T.\frac{t^3}{3} + \big((1-a)(1-b)\big)T^2.\frac{t^2}{2}\right]_0^{aT} \\ & + \frac{\lambda^3}{T}\left[\frac{t^3}{4} + ((-a)+(1-b))T.\frac{t^3}{3} + \big((-a)(1-b)\big)T^2.\frac{t^2}{2}\right]_0^{aT} \\ & + \frac{\lambda^3}{T}\left[\frac{t^3}{4} + ((-a)+(-b))T.\frac{t^3}{3} + \big((-a)(-b)\big)T^2.\frac{t^2}{2}\right]_{aT}^{bT} \end{aligned}$$

10.3 **Failure of 4 Items**

Consider the following graph



The joint probability of failure is given by

$F(t) = \lambda t(\lambda t + (1-a)\lambda T)(\lambda t + (1-b)\lambda T)(\lambda t + (1-c)\lambda T)$ [0,aT]
$F(t) = \lambda t(\lambda t + (-a)\lambda T)(\lambda t + (1-b)\lambda T)(\lambda t + (1-c)\lambda T)$ [aT,bT]
$F(t) = \lambda t(\lambda t + (-a)\lambda T)(\lambda t + (-b)\lambda T)(\lambda t + (1-c)\lambda T)$ [bT,cT]
$F(t) = \lambda t(\lambda t + (-a)\lambda T)(\lambda t + (-b)\lambda T)(\lambda t + (-c)\lambda T)$ [cT,T]

We can write this as: $F(t) = x(x+A)(x+B)(x+C)$
Where:

$x = \lambda t;$
$A = (1-a)\lambda T \; [t < aT]$
$A = (-a)\lambda T \; [t \geq aT]$
B= $(1-b)\lambda T \; [t < bT]$
B= $(-b)\lambda T \; [t \geq bT]$
$C = (1-c)\lambda T \; [t < cT]$
$C = (-c)\lambda T \; [t \geq cT]$

Expanding the expression for F(t)

$$F(t) = x(x^2 + (A+B)x + AB)(x+C)$$
$$F(t) = x(x^3 + (A+B+C)x^2 + (AB+AC+BC)x + ABC)$$
$$F(t) = x^4 + (A+B+C)x^3 + (AB+AC+BC)x^2 + ABCx$$

Note: The coefficients for powers of x (other than the first) are:

- the sum of all the solos, then
- the sum of all the pairs, then
- the sum of all the triples

This pattern is repeated for greater powers.

So, for 4 failures with staggered proof testing:

$$F(t) = \lambda^4 t^4 + (A+B+C)\lambda^3 t^3 + (AB+AC+BC)\lambda^2 t^2 + ABC\lambda t$$

and:

$$\begin{aligned} F_{AV} = {} & \frac{1}{T}\left[\lambda^4\frac{t^5}{5} + (A+B+C)\lambda^3\frac{t^4}{4} + (AB+AC+BC)\lambda^2\frac{t^3}{3} + ABC\lambda\frac{t^2}{2}\right]_0^{aT} \\ & + \frac{1}{T}\left[\lambda^4\frac{t^5}{5} + (A+B+C)\lambda^3\frac{t^4}{4} + (AB+AC+BC)\lambda^2\frac{t^3}{3} + ABC\lambda\frac{t^2}{2}\right]_{aT}^{bT} \\ & + \frac{1}{T}\left[\lambda^4\frac{t^5}{5} + (A+B+C)\lambda^3\frac{t^4}{4} + (AB+AC+BC)\lambda^2\frac{t^3}{3} + ABC\lambda\frac{t^2}{2}\right]_{bT}^{cT} \\ & + \frac{1}{T}\left[\lambda^4\frac{t^5}{5} + (A+B+C)\lambda^3\frac{t^4}{4} + (AB+AC+BC)\lambda^2\frac{t^3}{3} + ABC\lambda\frac{t^2}{2}\right]_{cT}^{T} \end{aligned}$$

Where:

$A = (1-a)\lambda T \; [t < aT]$
$A = (-a)\lambda T \quad [t \geq aT]$
B$= (1-b)\lambda T \; [t < bT]$
B$= (-b)\lambda T \quad [t \geq bT]$
$C = (1-c)\lambda T \; [t < cT]$
$C = (-c)\lambda T \quad [t \geq cT]$

We could simplify this whole expression by replacing as follows:

$A' = A/\lambda T$
$B' = B/\lambda T$
$C' = C/\lambda T$

Then, for 4 failures with staggered proof testing:

$$\begin{aligned} F_{AV} &= \frac{\lambda^4}{T}\left[\frac{t^5}{5} + (A' + B' + C')\frac{Tt^4}{4} + (A'B' + A'C + B'C')\frac{T^2t^3}{3} + A'B'C'\frac{T^3t^2}{2}\right]_0^{aT} \\ &+ \frac{\lambda^4}{T}\left[\frac{t^5}{5} + (A' + B' + C')\frac{Tt^4}{4} + (A'B' + A'C + B'C')\frac{T^2t^3}{3} + A'B'C'\frac{T^3t^2}{2}\right]_{aT}^{bT} \\ &+ \frac{\lambda^4}{T}\left[\frac{t^5}{5} + (A' + B' + C')\frac{Tt^4}{4} + (A'B' + A'C + B'C')\frac{T^2t^3}{3} + A'B'C'\frac{T^3t^2}{2}\right]_{bT}^{cT} \\ &+ \frac{\lambda^4}{T}\left[\frac{t^5}{5} + (A' + B' + C')\frac{Tt^4}{4} + (A'B' + A'C + B'C')\frac{T^2t^3}{3} + A'B'C'\frac{T^3t^2}{2}\right]_{cT}^{T} \end{aligned}$$

Changing variables and limits to simplify:

$$\begin{aligned} F_{AV} &= \lambda^4T^4\left[\frac{x^5}{5} + (A' + B' + C')\frac{x^4}{4} + (A'B' + A'C + B'C')\frac{x^3}{3} + A'B'C'\frac{x^2}{2}\right]_0^{a} \\ &+\lambda^4T^4\left[\frac{x^5}{5} + (A' + B' + C')\frac{x^4}{4} + (A'B' + A'C + B'C')\frac{x^3}{3} + A'B'C'\frac{x^2}{2}\right]_a^{b} \\ &+\lambda^4T^4\left[\frac{x^5}{5} + (A' + B' + C')\frac{x^4}{4} + (A'B' + A'C + B'C')\frac{x^3}{3} + A'B'C'\frac{x^2}{2}\right]_b^{c} \\ &+\lambda^4T^4\left[\frac{x^5}{5} + (A' + B' + C')\frac{x^4}{4} + (A'B' + A'C + B'C')\frac{x^3}{3} + A'B'C'\frac{x^2}{2}\right]_c^{1} \end{aligned}$$

Where:

$A' = (1-a) \; [x < a]$
$A' = (-a) \; [x \geq a]$
$B' = (1-b) \; [x < b]$

$B' = (-b)\ [x \geq b]$
$C' = (1 - c)\ [x < c]$
$C' = (-c)\ [x \geq c]$

Using this algorithm, we can calculate the F_{AV} for 4oo4 failures for any similar items with staggered proof tests.

10.4 **Failure of K Items**

It is possible to expand and find the general case from the above.

$$
\begin{aligned}
F_{AV} = {} & \left(\frac{\lambda T}{2}\right)^K 2^K \left[\frac{x^{K+1}}{K+1} + (Sum\ of\ solos)\frac{x^K}{K} + (Sum\ of\ pairs)\frac{x^{K-1}}{K-1} + (Sum\ of\ triples)\ \frac{x^{K-2}}{K-2} + \ldots.\right]_0^a \\
& + \left(\frac{\lambda T}{2}\right)^K 2^K \left[\frac{x^{K+1}}{K+1} + (Sum\ of\ solos)\frac{x^K}{K} + (Sum\ of\ pairs)\frac{x^{K-1}}{K-1} + (Sum\ of\ triples)\ \frac{x^{K-2}}{K-2} + \ldots.\right]_a^b \\
& + \cdots \\
& + \left(\frac{\lambda T}{2}\right)^K 2^K \left[\frac{x^{K+1}}{K+1} + (Sum\ of\ solos)\frac{x^K}{K} + (Sum\ of\ pairs)\frac{x^{K-1}}{K-1} + (Sum\ of\ triples)\ \frac{x^{K-2}}{K-2} + \ldots.\right]_{..}^1
\end{aligned}
$$

Where

$A' = (1 - a)\ [x < a]$
$A' = (-a)\ [x \geq a]$
$B' = (1 - b)\ [x < b]$
$B' = (-b)\ [x \geq b]$
$C' = (1 - c)\ [x < c]$
$C' = (-c)\ [x \geq c]$
etc. and $a < b < c < d \ldots \ldots . < 1$

10.5 **Calculating Coefficients**

In order to make use of this theory, we are looking for a matrix of coefficients $St_{N,K}$, to cover all the cases of N undetected failures out of M, where the coefficient can be used in calculations as a modifier in order that we can calculate the required PFD for N channels combined from the Nth power of the PFD for a single channel.

$$PFD_U^K = St_{L,K}(PFD_U^1)^K$$

Note: The sections above only cover the evaluation of F_{AV} (i.e. the average of the function) for a specific 'selection' of channels that are tested somewhere in the rotor.

We assume that $A', B', C'\ etc$ represent a subset of K items out of L where the proof testing of the L items is evenly spaced in time.

It can be seen then that $a, b, c\ etc.$ are rational numbers < 1.

Another process is required where all the possible subsets of K out of L are used to generate the F_{AV}.

The summation of the F_{AV} values for each subset is found and then divided by the number of subsets giving an F_{AV} which is now the average for K out of L.

A separate document called *Staggered Proof Testing Coefficients* shows how the programs carry out the calculations.

$St_{L,K}$ values are given below (for $L \leq 10$) where L is the row and K is the column)

	1	2	3	4	5	6	7	8	9	10
1	1.0000									
2	1.0000	0.8333								
3	1.0000	0.8889	0.6667							
4	1.0000	0.9167	0.7500	0.5229						
5	1.0000	0.9333	0.8000	0.6144	0.4053					
6	1.0000	0.9444	0.8333	0.6765	0.4938	0.3117				
7	1.0000	0.9524	0.8571	0.7215	0.5598	0.3917	0.2383			
8	1.0000	0.9583	0.8750	0.7555	0.6107	0.4558	0.3076	0.1814		
9	1.0000	0.9630	0.8889	0.7821	0.6511	0.5080	0.3666	0.2398	0.1376	
10	1.0000	0.9667	0.9000	0.8035	0.6840	0.5514	0.4170	0.2920	0.1858	0.1041

11 Synchronised Testing and Replacement

For cases where periodic proof testing of a sub-system is 'partial', the residual failures must be covered by thorough proof test, refurbishment or replacement in the longer term (in each of these cases, the maths is the same as a longer-term proof test of the residual failures.

In such circumstances, it is highly likely that the period for normal proof testing is synchronised with that of the longer-term test / refurbish / replace.

Synchronisation has a negative effect on the average performance. Where the difference in time periods is significant, it has less of an effect but, where time periods are similar, that is not the case.

In the general MooN modelling in *Fault Tolerant Systems*, a correction factor is required for when the periodicity is synchronised.

The requirements are to model the effect for failures in parallel, (some of which are du and some of which are dr).

In the following:

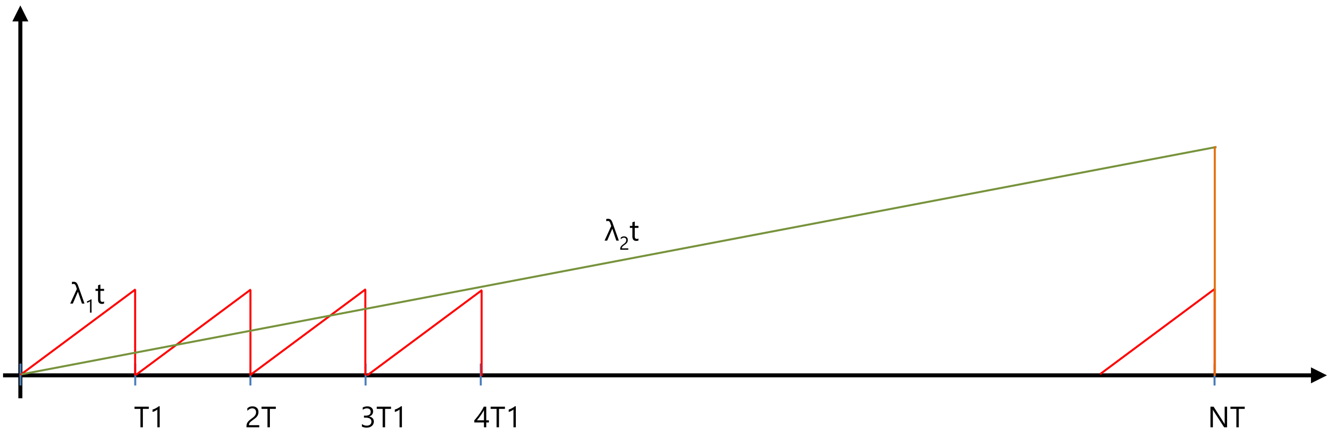
- the failure rate for the proof tested part (undiagnosed) is referred to as λ_1 with a test interval T_1;
- the failure rate for the residual part is referred to as λ_2 with a test interval T_2.

When there is more than on device, the joint probability includes x undiagnosed failures and y residual failures. It should be understood that if, for example $x = 2$, the shape of the corresponding curve is quadratic; if $x = 3$, the shape is cubic, etc.

11.1 **The joint probability of 2 linear functions ($\lambda_1\lambda_2$)**

11.1.1 **Finding the function**

We have to find the joint probability of failure of two devices and then find the average: the first device has failure rate $\lambda_1$and test interval T_1; the second has failure rate λ_2 and test interval T_2. Where $T_2 = NT_1$.



The joint probability of failure is given by

$F(t) = \lambda_1 t.\lambda_2 t = \lambda_1\lambda_2 t^2$ [0,T$_1$]

$F(t) = \lambda_1(t - T_1).\lambda_2 t = \lambda_1\lambda_2 t^2 - \lambda_1\lambda_2 T_1 t$ [T$_1$, 2T$_1$]

$F(t) = \lambda_1(t - 2T_1).\lambda_2 t = \lambda_1\lambda_2 t^2 - 2\lambda_1\lambda_2 T_1 t$ [2T$_1$, 3T$_1$]

$F(t) = \lambda_1(t - 3T_1).\lambda_2 t = \lambda_1\lambda_2 t^2 - 3\lambda_1\lambda_2 T_1 t$ [3T$_1$, 4T$_1$]

……

$F(t) = \lambda_1(t - (N-1)T_1).\lambda_2 t = \lambda_1\lambda_2 t^2 - (N-1)\lambda_1\lambda_2 T_1 t$ [(N-1)T$_1$, NT$_1$]

To find the average $F(t)$, we must first integrate over a time period and then divide by that period.

11.1.2 **Finding the average**

For interval [0,T$_1$], the integral is given by:

$$I(t) = \lambda_1\lambda_2 \int_0^{T_1} (t - 0T_1)t\ dt$$

For interval [0,T$_1$], the average is given by:

$$\frac{\lambda_1\lambda_2}{T_1}\left(\left[\frac{t^3}{3}\right]_{0T_1}^{1T_1} - (0)\left[\frac{T_1 t^2}{2}\right]_{0T_1}^{1T_1}\right)$$

$$= {\lambda_1}^1{\lambda_2}^1{T_1}^2\left(\left[\frac{(1^3 - 0^3)}{3}\right] - (0)\left[\frac{(1^2 - 0^2)}{2}\right]\right)$$

For interval [T$_1$, 2T$_1$], the average is given by:

$$= {\lambda_1}^1{\lambda_2}^1{T_1}^2\left(\left[\frac{(2^3 - 1^3)}{3}\right] - (1)\left[\frac{(2^2 - 1^2)}{2}\right]\right)$$

For interval [2T$_1$, 3T$_1$], the average is given by:

$$= {\lambda_1}^1{\lambda_2}^1{T_1}^2\left(\left[\frac{(3^3 - 2^3)}{3}\right] - (2)\left[\frac{(3^2 - 2^2)}{2}\right]\right)$$

For interval [(N-1)T$_1$, NT$_1$], the average is given by:

$$= {\lambda_1}^1{\lambda_2}^1{T_1}^2\left(\left[\frac{(N^3 - (N-1)^3)}{3}\right] - (N-1)\left[\frac{(N^2 - (N-1)^2)}{2}\right]\right)$$

Using differences to find the cumulation, we find it is given by:

$$\frac{1}{4}N^2 + \frac{1}{12}N$$

This is the accumulation of all N averages so the overall average is given by:

$${\lambda_1}^1{\lambda_2}^1{T_1}^2\left(\frac{1}{4}N^2 + \frac{1}{12}N\right)$$

$${\lambda_1}^1{\lambda_2}^1{T_1}^2\left(\frac{1}{4}N + \frac{1}{12}\right)$$

Note that: $T_1 = T_2/N$

Therefore, the average is given by:

$$\lambda_1{}^1\lambda_2{}^1T_1T_2\left(\frac{1}{4}+\frac{1}{12N}\right)$$

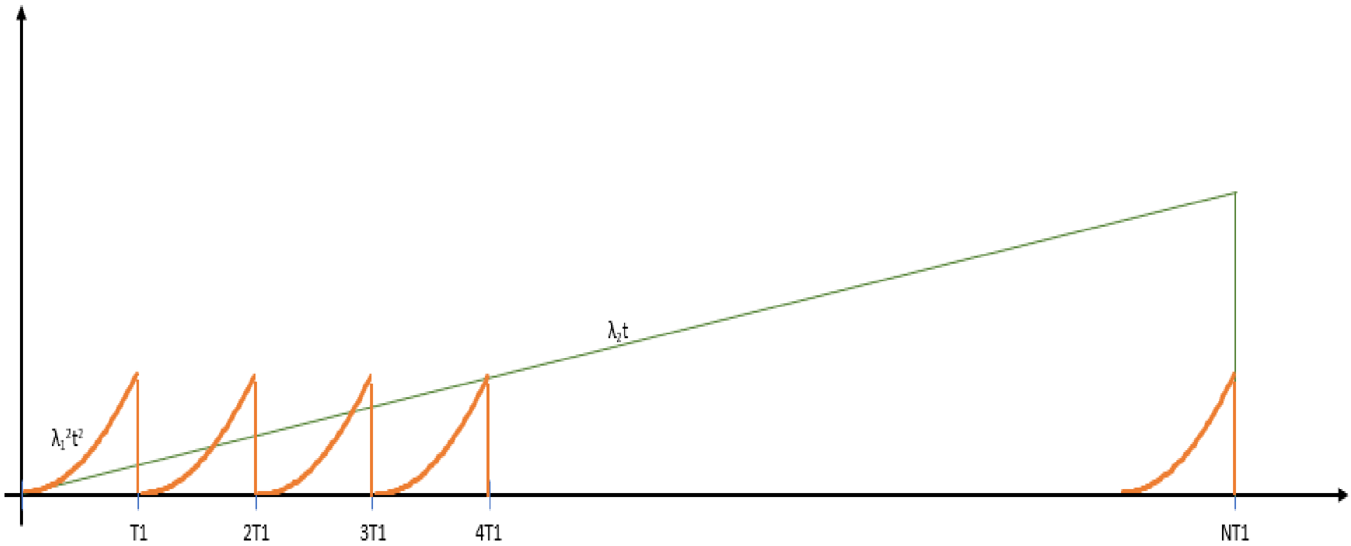
$$\frac{\lambda_1{}^1\lambda_2{}^1T_1T_2}{4}\left(1+\frac{1}{3N}\right)$$

The correction factor is: $\left(1+\frac{1}{3N}\right)$

11.2 The joint probability of a short quadratic and longer linear function ($\lambda_1{}^2\lambda_2$)

11.2.1 Finding the function

We have to find the joint probability of failure of three devices and then find the average: the first two device have a failure rate $\lambda_1$and test interval T_1; the third has failure rate $\lambda_2$and test interval T_2.
Where $T_2 = NT_1$.



The joint probability of failure is given by:

$F(t)=\lambda_1{}^2(t-0T_1)^2.\lambda_2 t=\lambda_1{}^2\lambda_2\left(t^3-2(0)T_1t^2+(0)^2T_1{}^2t\right)$	[0,T₁]
$F(t)=\lambda_1{}^2(t-1T_1)^2.\lambda_2 t=\lambda_1{}^2\lambda_2\left(t^3-2(1)T_1t^2+(1)^2T_1{}^2t\right)$	$[T_1, 2T_1]$
$F(t)=\lambda_1{}^2(t-2T_1)^2.\lambda_2 t=\lambda_1{}^2\lambda_2\left(t^3-2(2)T_1t^2+(2)^2T_1{}^2t\right)$	$[2T_1, 3T_1]$
$F(t)=\lambda_1{}^2(t-3T_1)^2.\lambda_2 t=\lambda_1{}^2\lambda_2\left(t^3-2(3)T_1t^2+(3)^2T_1{}^2t\right)$	$[3T_1, 4T_1]$
……	
$F(t)=\lambda_1{}^2(t-(N-1)T_1)^2.\lambda_2 t=\lambda_1{}^2\lambda_2\left(t^3-2(N-1)T_1t^2+(N-1)^2T_1{}^2t\right)$	$[(N-1)T_1, NT_1]$

11.2.2 Finding the average

For interval [0,T_1], the integral is given by:

$$I(t) = {\lambda_1}^2\lambda_2 \int_{0T_1}^{1T_1} t^3 - 2(0)T_1t^2 + (0)^2{T_1}^2t\,dt$$

For interval [0,T_1], the average is given by:

$$\frac{{\lambda_1}^2\lambda_2}{T_1}\left(\left[\frac{t^4}{4}\right]_{0T_1}^{1T_1} - 2(0)\left[\frac{T_1t^3}{3}\right]_{0T_1}^{1T_1} + (0)^2\left[\frac{{T_1}^2t^2}{2}\right]_{0T_1}^{1T_1}\right)$$

$$= {\lambda_1}^2\lambda_2{T_1}^3\left(\left[\frac{(1^4-0^4)}{4}\right] - 2(0)\left[\frac{(1^3-0^3)}{3}\right] + (0)^2\left[\frac{(1^2-0^2)}{2}\right]\right)$$

For interval [T_1,2T_1], the average is given by:

$$= {\lambda_1}^2\lambda_2{T_1}^3\left(\left[\frac{(2^4-1^4)}{4}\right] - 2(1)\left[\frac{(2^3-1^3)}{3}\right] + (1)^2\left[\frac{(2^2-1^2)}{2}\right]\right)$$

For interval [2T_1,3T_1], the average is given by:

$$= {\lambda_1}^2\lambda_2{T_1}^3\left(\left[\frac{(3^4-2^4)}{4}\right] - 2(2)\left[\frac{(3^3-2^3)}{3}\right] + (2)^2\left[\frac{(3^2-2^2)}{2}\right]\right)$$

For interval [(N-1)T_1,NT_1], the average is given by:

$$= {\lambda_1}^2\lambda_2{T_1}^3\left(\left[\frac{(N^4-(N-1)^4)}{4}\right] - 2(N-1)\left[\frac{(N^3-(N-1)^3)}{3}\right] + (N-1)^2\left[\frac{(N^2-(N-1)^2)}{2}\right]\right)$$

Using differences, we find the formula for the sum of averages over the N intervals is given by:

$$= {\lambda_1}^2\lambda_2{T_1}^3\left(\frac{1}{6}N^2 + \frac{1}{12}N\right)$$

The overall average is given by dividing this by N.
Overall average:

$${\lambda_1}^2\lambda_2{T_1}^3\left(\frac{1}{6}N + \frac{1}{12}\right)$$

Note that: $T_1 = T_2/N$
Therefore, the average is given by:

$${\lambda_1}^2{T_1}^2\lambda_2T_2\left(\frac{1}{6} + \frac{1}{12N}\right)$$

$$= \frac{{\lambda_1}^2{T_1}^2\lambda_2T_2}{6}\left(1 + \frac{1}{2N}\right)$$

If we had a pair of devices with failure rate of λ_1, we would expect a PFD_{AV} of $\frac{4}{3}\left(\frac{\lambda_1T_1}{2}\right)^2$
If we had a device with failure rate of λ_2, we would expect a PFD_{AV} of $\frac{\lambda_2T_2}{2}$

If these were all in parallel and the two were independent, we would expect a joint PFD_{AV} given by:

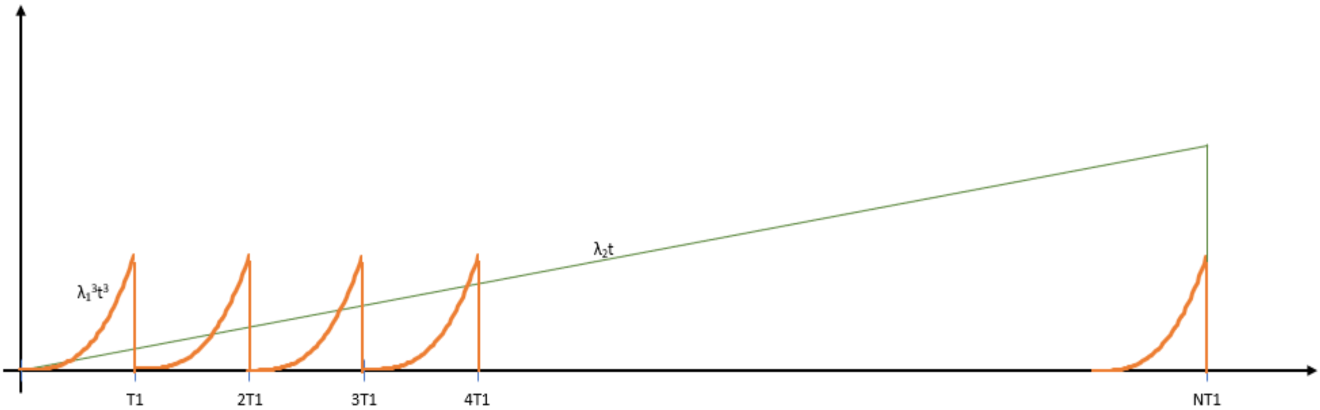
$$PFD_{AV} = \frac{4}{3}\left(\frac{\lambda_1 T_1}{2}\right)^2 \times \frac{\lambda_2 T_2}{2}$$
$$PFD_{AV} = \frac{{\lambda_1}^2 {T_1}^2 \lambda_2 T_2}{6}$$

The correction factor for a short quadratic tested function and a long linear replacement function is therefore $\left(1 + \frac{1}{2N}\right)$.

11.3 The joint probability of a short cubic and longer linear function ($\lambda_1^3\lambda_2$)

11.3.1 Finding the function

We have to find the joint probability of failure of four devices and then find the average: the first three device have a failure rate $\lambda_1$and test interval T_1; the fourth has failure rate $\lambda_2$and test interval T_2.
Where $T_2 = ZN$.



The joint probability of failure is given by:

$F(t) = {\lambda_1}^3(t - 0T_1)^3.\lambda_2 t = {\lambda_1}^3\lambda_2\left(t^4 - 3.(0)T_1 t^3 + 3.(0)^2{T_1}^2 t^2 - (0)^3{T_1}^3 t\right)$ [0,T$_1$]

$F(t) = {\lambda_1}^3(t - 1T_1)^3.\lambda_2 t = {\lambda_1}^3\lambda_2\left(t^4 - 3.(1)T_1 t^3 + 3.(1)^2{T_1}^2 t^2 - (1)^3{T_1}^3 t\right)$ [T$_1$, 2T$_1$]

$F(t) = {\lambda_1}^3(t - 2T_1)^3.\lambda_2 t = {\lambda_1}^3\lambda_2\left(t^4 - 3.(2)T_1 t^3 + 3.(2)^2{T_1}^2 t^2 - (2)^3{T_1}^3 t\right)$ [2T$_1$, 3T$_1$]

......

$F(t) = {\lambda_1}^3(t - (N-1)T_1)^3.\lambda_2 t = {\lambda_1}^3\lambda_2\left(t^4 - 3.(N-1)T_1 t^3 + 3.(N-1)^2{T_1}^2 t^2 - (N-1)^3{T_1}^3 t\right)$
[(N-1)T$_1$, NT$_1$]

11.3.2 Finding the average

For interval [0 T$_1$,1T$_1$], the integral is given by:

$$I(t) = {\lambda_1}^3\lambda_2 \int_{0T_1}^{1T_1} t^4 - 3.(0)T_1t^3 + 3.(0)^2{T_1}^2t^2 - (0)^3{T_1}^3t\,dt$$

For interval [0 T_1,1T_1], the average is given by:

$$\frac{{\lambda_1}^3\lambda_2}{T_1}\left(\left[\frac{t^5}{5}\right]_{0T_1}^{1T_1} - 3.(0)\left[\frac{T_1t^4}{4}\right]_{0T_1}^{1T_1} + 3.(0)^2\left[\frac{{T_1}^2t^3}{3}\right]_{0T_1}^{1T_1} - (0)^3\left[\frac{{T_1}^3t^2}{2}\right]_{0T_1}^{1T_1}\right)$$

$${\lambda_1}^3\lambda_2{T_1}^4\left(\left[\frac{(1^5-0^5)}{5}\right] - 3.(0)\left[\frac{(1^4-0^4)}{4}\right] + 3(0)^2\left[\frac{(1^3-0^3)}{3}\right] - (0)^3\left[\frac{(1^2-0^2)}{2}\right]\right)$$

For interval [1 T_1,2T_1], the average is given by:

$${\lambda_1}^3\lambda_2{T_1}^4\left(\left[\frac{(2^5-1^5)}{5}\right] - 3.(1)\left[\frac{(2^4-1^4)}{4}\right] + 3(1)^2\left[\frac{(2^3-1^3)}{3}\right] - (1)^3\left[\frac{(2^2-1^2)}{2}\right]\right)$$

For interval [2 T_1,3T_1], the average is given by:

$${\lambda_1}^3\lambda_2{T_1}^4\left(\left[\frac{(3^5-2^5)}{5}\right] - 3.(2)\left[\frac{(3^4-2^4)}{4}\right] + 3(2)^2\left[\frac{(3^3-2^3)}{3}\right] - (2)^3\left[\frac{(3^2-2^2)}{2}\right]\right)$$

Using differences, we find the formula for the sum of averages over the N intervals is given by:

$$= {\lambda_1}^3\lambda_2{T_1}^4\left(\frac{1}{8}N^2 + \frac{3}{40}N\right)$$

The overall average is given by dividing this by N.
Overall average:

$$\frac{{\lambda_1}^3\lambda_2{T_1}^4}{8}\left(1N + \frac{3}{5}\right)$$

Note that: $T_1 = T_2/N$
Overall average:

$$\frac{{\lambda_1}^3\lambda_2{T_1}^3T_2}{8}\left(1 + \frac{3}{5N}\right)$$

If we had three devices with failure rate of λ_1, we would expect a PFD_{AV} of $2\left(\frac{\lambda_1T_1}{2}\right)^3$

If we had one device with failure rate of λ_2, we would expect a PFD_{AV} of $\frac{\lambda_2T_2}{2}$

If these were all in parallel and the two were independent, we would expect a joint PFD_{AV} given by:

$$PFD_{AV} = 2\left(\frac{\lambda_1T_1}{2}\right)^3 \times \frac{\lambda_2T_2}{2}$$

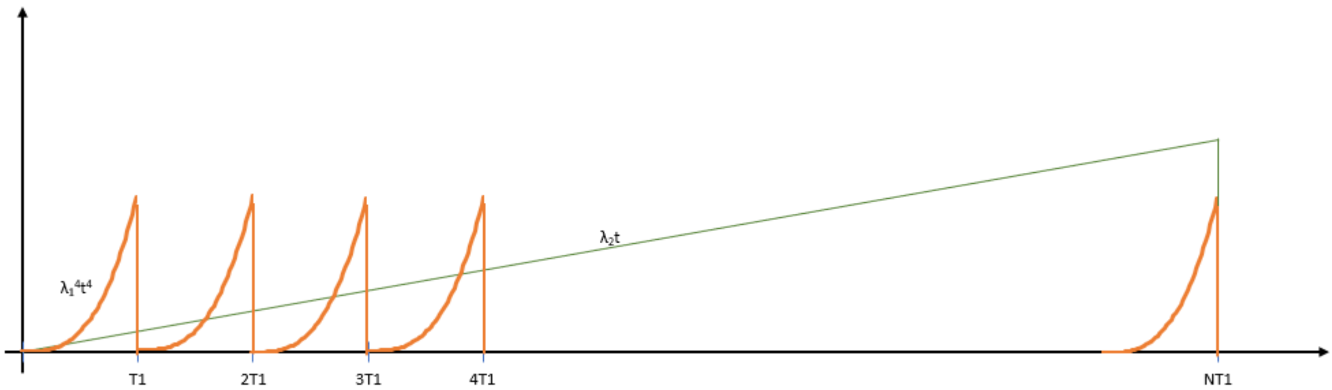
$$PFD_{AV} = \frac{1}{8}{\lambda_1}^3{T_1}^3\lambda_2T_2$$

The correction factor for a short cubic tested function and a long linear replacement function is therefore $\left(1 + \frac{3}{5N}\right)$.

11.4 The joint probability of a short quartic and longer linear function ($\lambda_1{}^4\lambda_2$)

11.4.1 Finding the function

We have to find the joint probability of failure of five devices and then find the average: the first four device have a failure rate $\lambda_1$and test interval T_1; the fifth has failure rate $\lambda_2$and test interval T_2.
Where $T_2 = NT_1$.



The joint probability of failure is given by:
$F(t) = \lambda_1{}^4(t - 0T_1)^4.\lambda_2 t = \lambda_1{}^4\lambda_2\left(t^5 - 4.(0)T_1t^4 + 6.(0)^2T_1{}^2t^3 - 4.(0)^3T_1{}^3t^2 + (0)^4T_1{}^4t\right)$[0,$T_1$]
$F(t) = \lambda_1{}^4(t - 1T_1)^4.\lambda_2 t = \lambda_1{}^4\lambda_2\left(t^5 - 4.(1)T_1t^4 + 6.(1)^2T_1{}^2t^3 - 4.(1)^3T_1{}^3t^2 + (1)^4T_1{}^4t\right)$[$T_1$, 2$T_1$]
$F(t) = \lambda_1{}^4(t - 2T_1)^4.\lambda_2 t = \lambda_1{}^4\lambda_2\left(t^5 - 4.(2)T_1t^4 + 6.(2)^2T_1{}^2t^3 - 4.(2)^3T_1{}^3t^2 + (2)^4T_1{}^4t\right)$[2$T_1$, 3$T_1$]
……
$F(t) = \lambda_1{}^4(t - (N-1)T_1)^4.\lambda_2 t = \lambda_1{}^4\lambda_2\left(t^5 - 4.(N-1)T_1t^4 + 6.(N-1)^2T_1{}^2t^3 - 4.(N-1)^3T_1{}^3t^2 + (N-1)^4T_1{}^4t\right)$ [(N-1)T_1, NT_1]

11.4.2 Finding the average

For interval [0 T_1,1T_1], the integral is given by:

$$I(t) = \lambda_1{}^4\lambda_2 \int_{0T_1}^{1T_1} t^5 - 4.(0)T_1t^4 + 6.(0)^2T_1{}^2t^3 - 4.(0)^3T_1{}^3t^2 + (0)^4T_1{}^4t\,dt$$

For interval [0T_1,1T_1], the average is given by:

$$\frac{\lambda_1{}^4\lambda_2}{T_1}\left(\left[\frac{t^6}{6}\right]_{0T_1}^{1T_1} - 4.(0)\left[\frac{T_1t^5}{5}\right]_{0T_1}^{1T_1} + 6.(0)^2\left[\frac{T_1{}^2t^4}{4}\right]_{0T_1}^{1T_1} - 4.(0)^3\left[\frac{T_1{}^3t^3}{3}\right]_{0T_1}^{1T_1} + (0)^4\left[\frac{T_1{}^4t^2}{2}\right]_{0T_1}^{1T_1}\right)$$

$$\lambda_1{}^4\lambda_2T_1{}^5\left(\left[\frac{(1^6 - 0^6)}{6}\right] - 4.(0)\left[\frac{(1^5 - 0^5)}{5}\right] + 6(0)^2\left[\frac{(1^4 - 0^4)}{4}\right] - 4.(0)^3\left[\frac{(1^3 - 0^3)}{3}\right] + (0)^4\left[\frac{(1^2 - 0^2)}{2}\right]\right)$$

For interval [1T_1,2T_1], the average is given by:

$$\lambda_1{}^4\lambda_2 T_1{}^5\left(\left[\frac{(2^6-1^6)}{6}\right]-4.(1)\left[\frac{(2^5-1^5)}{5}\right]+6(1)^2\left[\frac{(2^4-1^4)}{4}\right]-4.(1)^3\left[\frac{(2^3-1^3)}{3}\right]+(1)^4\left[\frac{(2^2-1^2)}{2}\right]\right)$$

For interval [2 T_1,3T_1], the average is given by:

$$\lambda_1{}^4\lambda_2 T_1{}^5\left(\left[\frac{(3^6-2^6)}{6}\right]-4.(2)\left[\frac{(3^5-2^5)}{5}\right]+6(2)^2\left[\frac{(3^4-2^4)}{4}\right]-4.(2)^3\left[\frac{(3^3-2^3)}{3}\right]+(2)^4\left[\frac{(3^2-2^2)}{2}\right]\right)$$

Using differences, we find the formula for the sum of averages over the N intervals is given by:

$$=\lambda_1{}^4\lambda_2 T_1{}^5\left(\frac{1}{10}N^2+\frac{1}{15}N\right)$$

The overall average is given by dividing this by N.
Overall average:

$$\frac{\lambda_1{}^4\lambda_2 T_1{}^5}{10}\left(1N+\frac{2}{3}\right)$$

Note that: $T_1=T_2/N$
Overall average:

$$\frac{\lambda_1{}^4\lambda_2 T_1{}^4 T_2}{10}\left(1+\frac{2}{3N}\right)$$

If we had four devices with failure rate of λ_1, we would expect a PFD_{AV} of $\frac{16}{5}\left(\frac{\lambda_1 T_1}{2}\right)^4$
If we had one device with failure rate of λ_2, we would expect a PFD_{AV} of $\frac{\lambda_2 T_2}{2}$
If these were all in parallel and the two were independent, we would expect a joint PFD_{AV} given by:

$$PFD_{AV}=\frac{16}{5}\left(\frac{\lambda_1 T_1}{2}\right)^4\times\frac{\lambda_2 T_2}{2}$$

$$PFD_{AV}=\frac{1}{10}\lambda_1{}^4 T_1{}^4\lambda_2 T_2$$

The correction factor for a short quartic tested function and a long linear replacement function is therefore $\left(1+\frac{2}{3N}\right)$.

11.5 **Joint probability of short kth-ic and long linear**

From the preceding number patterns we can see the general 'correction' factor for K tested elements in parallel with one longer term replaced element.
We have already compensated for the fact that testing is synchronised but we are trying to find the effect where that is also synchronised with replacement where: $T_1=T_2/N$.

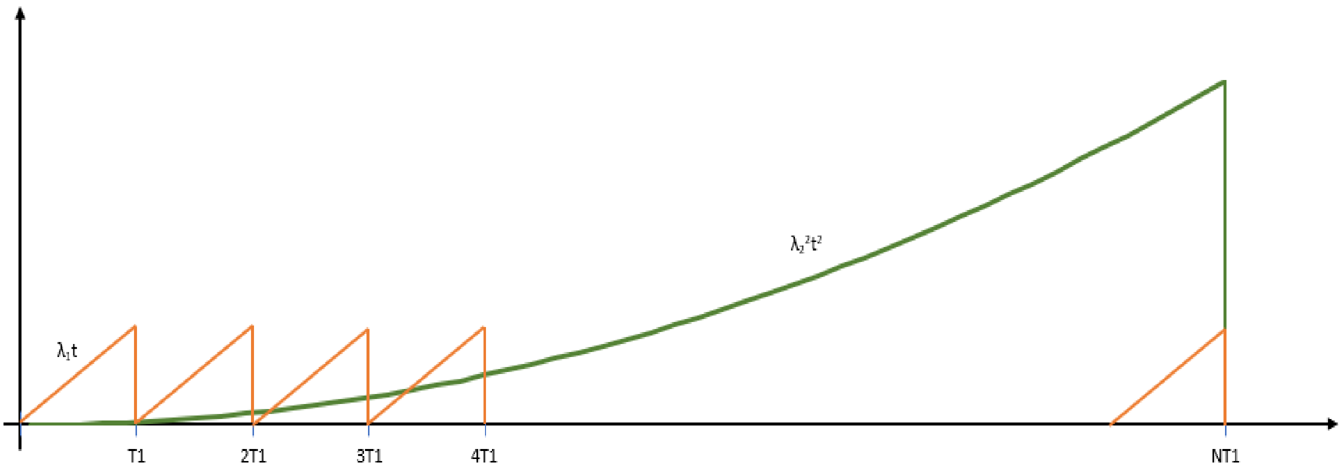
The correction factors are as follows

1	$1 + \frac{1}{3N}$
2	$1 + \frac{2}{4N}$
3	$1 + \frac{3}{5N}$
4	$1 + \frac{4}{6N}$
…….	
K	$1 + \frac{K}{(K+2)N}$

11.6 The joint probability of a short linear and longer quadratic function ($\lambda_1\lambda_2^2$)

11.6.1 Finding the function

We have to find the joint probability of failure of three devices and then find the average: the first device has a failure rate $\lambda_1$and test interval T_1; the remaining pair have a failure rate λ_2 and test interval T_2.
Where $T_2 = NT_1$.



$F(t) = \lambda_1 t \lambda_2{}^2 t^2 = \lambda_1 \lambda_2{}^2 t^3$ [0,T_1]
$F(t) = \lambda_1 (t - T_1) \lambda_2{}^2 t^2 = \lambda_1 \lambda_2{}^2 t^3 - \lambda_1 \lambda_2{}^2 T_1 t^2$ [T_1, $2T_1$]
$F(t) = \lambda_1 (t - 2T_1) \lambda_2{}^2 t^2 = \lambda_1 \lambda_2{}^2 t^3 - 2\lambda_1 \lambda_2{}^2 T_1 t^2$ [$2T_1$, $3T_1$]
$F(t) = \lambda_1 (t - 3T_1) \lambda_2{}^2 t^2 = \lambda_1 \lambda_2{}^2 t^3 - 3\lambda_1 \lambda_2{}^2 T_1 t^2$ [$3T_1$, $4T_1$]

11.6.2 Finding the average

For interval [0 T_1,1T_1], the integral is given by:

$$I(t) = \lambda_1 \lambda_2{}^2 \int_{0T_1}^{1T_1} t^3 - (0)T_1 t^2 \, dt$$

For interval [0 T_1,1T_1], the average is given by:

$$\frac{\lambda_1 {\lambda_2}^2}{T_1}\left(\left[\frac{t^4}{4}\right]_{0T_1}^{1T_1} - (0)\left[\frac{T_1 t^3}{3}\right]_{0T_1}^{1T_1}\right)$$

$$\lambda_1 {\lambda_2}^2 {T_1}^3\left(\left[\frac{(1^4 - 0^4)}{4}\right] - (0)\left[\frac{(1^3 - 0^3)}{3}\right]\right)$$

For interval [1 T_1,2T_1], the average is given by:

$$\lambda_1 {\lambda_2}^2 {T_1}^3\left(\left[\frac{(2^4 - 1^4)}{4}\right] - (1)\left[\frac{(2^3 - 1^3)}{3}\right]\right)$$

For interval [2 T_1,3T_1], the average is given by:

$$\lambda_1 {\lambda_2}^2 {T_1}^3\left(\left[\frac{(3^4 - 2^4)}{4}\right] - (2)\left[\frac{(3^3 - 2^3)}{3}\right]\right)$$

Using differences, we find the formula for the sum of averages over the N intervals is given by:

$$= \lambda_1 {\lambda_2}^2 {T_1}^3\left(\frac{1}{6}N^3 + \frac{1}{12}N^2\right)$$

The overall average is given by dividing this by N.
Overall average:

$$= \lambda_1 {\lambda_2}^2 {T_1}^3\left(\frac{1}{6}N^2 + \frac{1}{12}N\right)$$

Note that: $T_1 = T_2/N$
Overall average:

$$= \lambda_1 {\lambda_2}^2 T_1 {T_2}^2\left(\frac{1}{6} + \frac{1}{12N}\right)$$

$$= \frac{\lambda_1 {\lambda_2}^2 T_1 {T_2}^2}{6}\left(1 + \frac{1}{2N}\right)$$

If we had one device with failure rate of λ_1, we would expect a PFD_{AV} of $\frac{\lambda_1 T_1}{2}$

If we had two devices with failure rate of λ_2, we would expect a PFD_{AV} of $\frac{4}{3}\left(\frac{\lambda_2 T_2}{2}\right)^2$

If these were all in parallel and the two were independent, we would expect a joint PFD_{AV} given by:

$$PFD_{AV} = \frac{\lambda_1 T_1}{2} \times \frac{4}{3}\left(\frac{\lambda_2 T_2}{2}\right)^2$$

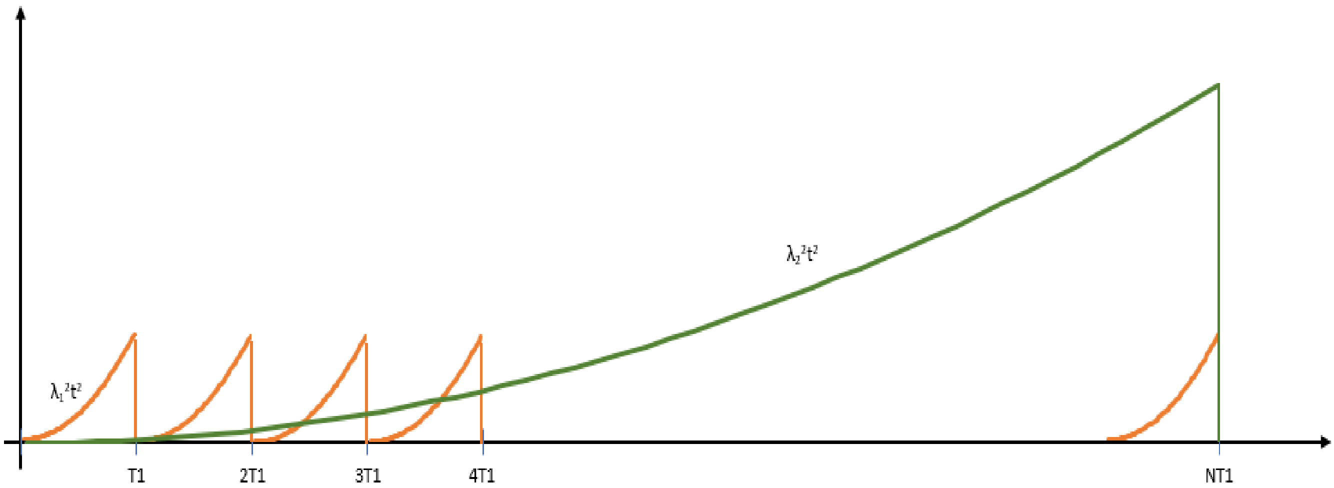
$$PFD_{AV} = \frac{1}{6}\lambda_1 T_1 {\lambda_2}^2 {T_2}^2$$

The correction factor for a short linear tested function and a long quadratic replacement function is therefore $\left(1 + \frac{1}{2N}\right)$.

11.7 The joint probability of two quadratics ($\lambda_1^2\lambda_2^2$)

11.7.1 Finding the function

We have to find the joint probability of failure of four devices and then find the average: the first two devices have a failure rate λ_1 and test interval T_1; the second two have a failure rate λ_2 and test interval T_2.
Where $T_2 = NT_1$.



The joint probability of failure is given by

$F(t) = {\lambda_1}^2{\lambda_2}^2(t - 0T_1)^2t^2 = {\lambda_1}^2{\lambda_2}^2\left(t^4 - 2(0)T_1t^3 + (0)^2{T_1}^2t^2\right)$ [0,T₁]

$F(t) = {\lambda_1}^2{\lambda_2}^2(t - 1T_1)^2t^2 = {\lambda_1}^2{\lambda_2}^2\left(t^4 - 2(1)T_1t^3 + (1)^2{T_1}^2t^2\right)$ [T_1, $2T_1$]

$F(t) = {\lambda_1}^2{\lambda_2}^2(t - 2T_1)^2t^2 = {\lambda_1}^2{\lambda_2}^2\left(t^4 - 2(2)T_1t^3 + (2)^2{T_1}^2t^2\right)$ [$2T_1$, $3T_1$]

$F(t) = {\lambda_1}^2{\lambda_2}^2(t - 3T_1)^2t^2 = {\lambda_1}^2{\lambda_2}^2\left(t^4 - 2(3)T_1t^3 + (3)^2{T_1}^2t^2\right)$ [$3T_1$, $4T_1$]

11.7.2 Finding the Average

For interval [0 T_1,1T_1], the integral is given by:

$$I(t) = {\lambda_1}^2{\lambda_2}^2 \int_{0T_1}^{1T_1} t^4 - 2(0)T_1t^3 + (0)^2{T_1}^2t^2dt$$

For interval [0 T_1,1T_1], the average is given by:

$$\frac{{\lambda_1}^2{\lambda_2}^2}{T_1}\left(\left[\frac{t^5}{5}\right]_{0T_1}^{1T_1} - 2(0)\left[\frac{T_1t^4}{4}\right]_{0T_1}^{1T_1} + (0)^2\left[\frac{{T_1}^2t^3}{3}\right]_{0T_1}^{1T_1}\right)$$
$$= {\lambda_1}^2{\lambda_2}^2{T_1}^4\left(\left[\frac{(1^5 - 0^5)}{5}\right] - 2(0)\left[\frac{(1^4 - 0^4)}{4}\right] + (0)^2\left[\frac{(1^3 - 0^3)}{3}\right]\right)$$

For interval [1T_1,2T_1], the average is given by:

$$= {\lambda_1}^2{\lambda_2}^2{T_1}^4\left(\left[\frac{(2^5 - 1^5)}{5}\right] - 2(1)\left[\frac{(2^4 - 1^4)}{4}\right] + (1)^2\left[\frac{(2^3 - 1^3)}{3}\right]\right)$$

For interval [$2T_1$,$3T_1$], the average is given by:

$$= {\lambda_1}^2{\lambda_2}^2{T_1}^4\left(\left[\frac{(3^5-2^5)}{5}\right]-2(2)\left[\frac{(3^4-2^4)}{4}\right]+(2)^2\left[\frac{(3^3-2^3)}{3}\right]\right)$$

Using differences, we find the formula for the sum of averages over the N intervals is given by:

$$= {\lambda_1}^2{\lambda_2}^2{T_1}^4\left(\frac{1}{9}N^3+\frac{1}{12}N^2+\frac{1}{180}N\right)$$

The overall average is given by dividing this by N.
Overall average:

$${\lambda_1}^2{\lambda_2}^2{T_1}^4\left(\frac{1}{9}N^2+\frac{1}{12}N+\frac{1}{180}\right)$$

Note that: $T_1 = T_2/N$
Overall average:

$${\lambda_1}^2{\lambda_2}^2{T_1}^2{T_2}^2\left(\frac{1}{9}+\frac{1}{12N}+\frac{1}{180N^2}\right)$$

$$\frac{{\lambda_1}^2{\lambda_2}^2{T_1}^2{T_2}^2}{9}\left(1+\frac{3}{4N}+\frac{1}{20N^2}\right)$$

If we had a pair of devices with failure rate of λ_1, we would expect a PFD_{AV} of $\frac{4}{3}\left(\frac{\lambda_1 T_1}{2}\right)^2$
If we had a pair of devices with failure rate of λ_2, we would expect a PFD_{AV} of $\frac{4}{3}\left(\frac{\lambda_2 T_2}{2}\right)^2$
If these were all in parallel and the two were independent, we would expect a joint PFD_{AV} given by:

$$PFD_{AV} = \frac{4}{3}\left(\frac{\lambda_1 T_1}{2}\right)^2 \times \frac{4}{3}\left(\frac{\lambda_2 T_2}{2}\right)^2$$

If N =1, then $T_2 = T_1$ and:

$$PFD_{AV} = \frac{1}{9}{\lambda_1}^2{T_1}^2{\lambda_2}^2{T_2}^2$$

The correction factor for a quadratic tested function and a quadratic replacement function is therefore $\left(1+\frac{3}{4N}+\frac{1}{20N^2}\right)$.

11.8 **Automating the Process**

11.8.1 **The approach**

It can be seen from the foregoing that there is an algorithmic approach to determination of the formulae.

If we have x items at λ_1, T_1 and have y items at λ_2, T_2 which are tested synchronously and that the tests themselves are synchronised, we wish to develop a correction factor to the individual PFDs (assuming that they have already been corrected for synchronous testing). In other words, we are looking for the additional correction factor for if the two different tests are also synchronised.

We represent the above general case as ${\lambda_1}^x, {\lambda_2}^x$. It is implicit is that T_1 and T_2 apply respectively. In anticipation of using this in *Fault Tolerant* Systems, instead of using N, we will use another letter.
Let $T_2 = K.T_1$.

Let the order of the polynomial be w. We can see from the above that $w = \mathrm{x} + \mathrm{y} + 1$.
We can also see that the final order is reduced by x.

The following steps are then necessary

1. Develop the series for the integral in each interval. Let the interval number be N.

$$Interval\ Integral = C_0^N \frac{(N^w - (N-1)^w}{w} - C_1^N \frac{(N^{(w-1)} - (N-1)^{(w-1)}}{w-1} \ldots \ldots . C_x^N \frac{(N^{(w-x)} - (N-1)^{(w-x)}}{w-x}$$

$$Interval\ Integral = \sum_{i=0}^{x} \left(C_i^N (-1)^i \frac{(N^{(w-i)} - (N-1)^{(w-i)}}{w-i} \right)$$

2. Summate the $Interval\ Integrals$
3. Use differences techniques to find a formula for the polynomial based on the integer K
4. Use the same coefficients but divide through by K^x giving the first term with no power of K and the xth term of power K^{-x}.
5. Multiply through by $x + 1$ and $y + 1$. This corrects for the correction that would already have been applied for the individually synchronous PFDs and results in the first coefficient of 1.

The above was carried out using a spreadsheet for up to 10 components. The spreadsheet is called *Differences* because the method of determining the resultant polynomials used a ‘differences’ approach.

Differences.xlsx

The correction factor is always of the form:

$$K_0 N^0 + K_1 N^{-1} + K_2 N^{-2} + \cdots$$

In the following tables, empty cells have the implicit values of 0.

11.8.2 **The Findings**

11.8.2.1 λ_2=1

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	1/3								
2	1	1/2								
3	1	3/5								
4	1	2/3								
5	1	5/7								
6	1	3/4								
7	1	7/9								
8	1	4/5								
9	1	9/11								

11.8.2.2 λ_2=2

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	1/2	0							
2	1	3/4	1/20							
3	1	8/9	1/10							
4	1	1	1/7							
5	1	15/14	5/28							
6	1	9/8	5/24							
7	1	7/6	7/5							
8	1	6/5	27/106							

11.8.2.3 λ_2=3

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	2/3	0	-1/15						
2	1	1	1/10	-1/10						
3	1	6/5	1/5	-1/9						
4	1	4/3	2/7	-1/8						
5	1	10/7	5/14	-72/605						

6	1	13/12	5/12	-7/60						
7	1	14/9	7/15	-56/495						

11.8.2.4 λ_2=4

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	5/6	0	-1/6	0					
2	1	5/4	1/6	-1/4	-1/42					
3	1	3/2	1/3	-2/7	-1/21					
4	1	5/3	10/21	-25/84	-50/741					
5	1	25/14	25/42	-5/14	-20/247					
6	1	15/8	25/36	-7/24	-7/73					

11.8.2.5 λ_2=5

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	1	0	-1/3	0	01/21				
2	1	3/2	1/4	-1/2	-1/14	1/14				
3	1	9/5	1/2	-4/7	-1/7	48/593				
4	1	2	5/7	-3/5	-120/593	1/12				
5	1	15/7	8/9	-3/5	-1/4	6/73				

11.8.2.6 λ_2=6

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	7/6	0	-4/7	0	1/6	0			
2	1	7/4	7/20	-7/8	-1/6	1/4	1/40			
3	1	21/10	7/10	-1	-1/3	17/60	1/20			
4	1	7/3	1	-25/24	-17/36	7/24	7/99			

11.8.2.7 λ_2=7

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	4/3	0	-14/15	0	4/9	0	-1/15		
2	1	2	7/15	-7/5	-1/3	2/3	1/10	-1/10		
3	1	12/5	14/15	-8/5	-2/3	34/45	1/5	-32/283		

11.8.2.8 λ_2=8

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	3/2	0	-7/5	0	1	0	-3/10	0	
2	1	9/4	3/5	-21/10	-3/5	3/2	3/10	-9/20	-1/22	

11.8.2.9 λ_2=9

λ_1	K_0	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1	1	5/3	0	-2	0	2	0	-1	0	5/33

Note: Each of the coefficients in the tables above apply but $N > 1$ so anything beyond N^{-3} is ignored.

11.8.3 **Formulae**

The formulae for K as a function of x and y are derived from the above tables in the *differences* spreadsheet.
Where x is the index applied to λ_1 and y is the index applied to λ_2.

11.8.3.1 K_0

$$K_0 = 1$$

11.8.3.2 K_1

$$K_1 = \frac{x(y+1)}{2x+4}$$

11.8.3.3 K_2

$x = 1$: $K_2 = 0$
$y = 1$: $K_2 = 0$
x≠1, y≠1:

$$K_2 = (-4.847e^{-4}x^2 + 1.047e^{-2}x - 1.054e^{-2})(y^2 + y)$$

12 References

1. IEC 61508, Functional safety of electrical / electronic / programmable electronic (E/E/PE) safety related systems, Parts 1-7, 2010 (includes EN and BS EN variants).
2. IEC 61511, *Functional safety – Safety instrumented systems for the process industry sector*, Parts 1-3, 2017 (includes A1:2017 and the EN and BS EN variants).
3. ISA-TR84.00.02-2002 Part 1, Safety Instrumented Functions (SIF) – Safety Integrity Level (SIL) Evaluation Techniques – Part 1: Introduction, 2002.
4. ISA-TR84.00.02-2002 Part 2, Safety Instrumented Functions (SIF) – Safety Integrity Level (SIL) Evaluation Techniques – Part 2: Determining the SIL of a SIF via Simplified Equations, 2002.
5. VDI/VDE 2180 Part 3, Functional safety in the process industry – Verification of probability of failure on demand (PFD), 2019.
6. SINTEF A11612, Unrestricted Report – Use of the PDS Method for Railway Applications, June 2009.
7. Reliability Maintainability and Risk (10th Edition) – Dr David J Smith.
8. New approach to SIL verification – Mirek Generowicz of I&E Systems Pty – Australia (available free to download from *The 61508 Association* website).

13 Conclusion

This paper is part of a series of documents (see introduction) and therefore a conclusion is not required at this point.

14 Existing and Emerging Standards

IEC 61508:2010 (Series of standards, Edition 2).
IEC 61511-1:2017+A1:2017 (Edition 2).

15 61508 Association Recommended Practices

This document sets out to describe current best practices in reliability for functional safety systems, but does not seek to prescribe specific measures, since these will depend on the application and any existing constraints of the installation.

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